University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 6

1. *Relativistic transformation of a particle's polar angle.* Consider the usual Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with spatial origins coincident at $t = t' = 0$. As usual, frame $\mathcal{S}'$ moves in the $\hat{x}$ or $\hat{x}'$ direction with velocity βc with respect to frame $\mathcal{S}$. A particle is emitted by a radioactive source that is at rest with respect to $\mathcal{S}'$. As seen by an observer in $\mathcal{S}'$, the particle travels with velocity $\beta' c$ at an angle θ' with respect to the $\hat{x}'$ direction. However, as seen by an observer who is at rest with respect to the frame $\mathcal{S}$, prove that the particle makes a different angle θ with respect to the $\hat{x}$ direction, where

$$\tan\theta = \frac{\sin\theta'}{\gamma\big(\cos\theta' + (\beta/\beta')\big)} \ .$$

2. and **3.** (double credit problem)
Violation of time-reversal invariance was discovered in 1964 in the weak decay

$$K_L^0 \to \pi^+\pi^- \ ,$$

where the K_L^0 and $\pi^\pm$ are quark-antiquark pairs (including a strange quark in the K_L^0 case); a kaon has $\approx \frac{7}{2}$ of a pion's mass. In its own rest frame, the (spin 0) kaon decays isotropically. Suppose that the kaons compose a finite beam whose momentum per particle is $2m_K c$ ($\approx$ 1 GeV/c). With respect to the beam direction, find the laboratory angle θ at which the flux of decay pions per unit solid angle, $dN/d\Omega dt$, is *infinite*. [*Hint*: the answer is *not* $\theta = 90°$.]

4. Define the contravariant four-vectors

$$\begin{aligned}
A^\mu &\equiv \{V/c, \mathbf{A}\}\\
J^\mu &\equiv \{c\rho, \mathbf{J}\}\\
p^\mu &\equiv \{E/c, \mathbf{p}\}\\
k^\mu &\equiv \{\omega/c, \mathbf{k}\}\\
\partial^\mu &\equiv \{\partial/c\partial t, -\nabla\} \ .
\end{aligned}$$

Use the convention that repeated Greek indices are summed from 0 to 3. Employing primarily contravariant four-vectors, but making use of covariant four-vectors where appropriate, write a manifestly Lorentz invariant equation that is equivalent to
(**a**) the generalized de Broglie relation.
(**b**) conservation of electric charge.
(**c**) the Lorentz gauge condition.
(**d**) the wave equation, including sources, for the electromagnetic potentials in Lorentz gauge.

5. An object a^μ is a (contravariant) four-vector if it transforms (between frames as defined in Problem 1) according to

$$a'^\mu = \Lambda^\mu_\nu a^\nu \ ,$$

where Λ is the (symmetric) 4×4 Lorentz transformation matrix. (Conventionally, the superscript labels the row and the subscript labels the column, but this makes no difference for a symmetric matrix.) Covariant four-vectors instead transform according to

$$a'_\mu = (\Lambda^{-1})^\nu_\mu a_\nu$$

(otherwise the scalar product $a_\mu a^\mu = a'_\mu a'^\mu$ would not remain invariant for different Lorentz frames). Consider now an (arbitrary) four-tensor $H^{\mu\nu}$. In frame $\mathcal{S}$, $H^{\mu\nu}$ contracts with covariant four-vector a_ν to yield contravariant four-vector b^μ, according to

$$b^\mu = H^{\mu\nu} a_\nu \ .$$

In the frame $\mathcal{S}'$, requiring $H^{\mu\nu}$ to satisfy the transformation properties of a four-tensor, we define $H'^{\mu\nu}$ so that

$$b'^\mu = H'^{\mu\nu} a'_\nu \ .$$

Prove that

$$H'^{\mu\nu} = \Lambda^\mu_\rho H^{\rho\sigma} \Lambda^\nu_\sigma \ .$$

This defines the Lorentz transformation property of a four-tensor.

6. Consider the antisymmetric *electromagnetic field strength tensor*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ .$$

Prove that $F^{\mu\nu}$ is a four-tensor, *i.e.* it transforms according to the results of Problem 5.

7. Using the definitions of ∂^μ and A^μ, show by explicit calculation, element by element, that the covariant electromagnetic field strength tensor is equal to

$$F = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix} .$$

(The sign of this result is opposite to that of Griffiths; this is expected from his use of a metric tensor with sign opposite to the standard.)

8. Prove that the equation

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

is equivalent (in vacuum) to the two Maxwell equations which involve sources. (The two source-free Maxwell equations are already required to be true by the definition of A^μ.)